Estimating Risk of Failure of Engineering Structures using Predictive Likelihood

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Abstract

It has been common engineering practice to define characteristic values for loading and capacity of structures in order to assess the structural capacity of existing structures.

This approach, yet practical and intuitive, lead to the comparison of deterministic values (characteristic values) that had to represent all the variability of the problem and is considered to be conservative, as usually loading is overestimated and capacity underestimated, yielding to calculations with high safety margins for the extreme events.

Probabilistic methods have tried to overcome this limitation by computing the overall probability of failure (p_f) for the lifetime of the structure, taking into account the real probabilistic distribution of both loading and resistance.

In this paper, Predictive Likelihood (PL) is presented as a powerful method to determine the lifetime distribution for loading and resistance. From these lifetime distributions the probability of failure is computed.

An example of the application of the proposed method is finally presented. The result obtained using PL is then compared with the numerical approximation for the exact lifetime probability of failure.

Keywords: Bridge, Predictive Likelihood, Risk, Simulation, Statistics.

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INTRODUCTION

In its most basic approximation, Structural Reliability (SR) aims to provide an estimate of the probability of failure (p_f) of a given structural element. Given the stochastic distribution of all the variables, the p_f is considered to be the 'sum' of the failure probabilities over all the cases of resistance and load for which the load effect (*S*) (stress, bending moment, shear, etc.) exceeds the structural capacity (*R*) to resist the applied effect Eq. (1). In other words, any structural element is considered to have failed if its resistance *R* is less than the stress resultant *S* acting on it. The probability of failure will then be defined as the number of failures over the total number of outcomes. As shown by Melchers [1]

$$p_f = P(R - S \leq 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{s \geq r} f_R(r) f_S(s)\, drds = P(Z \leq 0) \qquad \text{Eq. (1)}$$

where $f_R(\cdot)$ represents the probability density function (PDF) of the capacity and $f_S(\cdot)$ the PDF of the loading.

For the special, but common case when loading and resistance are independent, Eq. (1) can be expressed as

$$p_f = P(R - S \leq 0) = \int_{-\infty}^{\infty} F_R(s)\, f(s)\, ds = P(Z \leq 0) \qquad \text{Eq. (2)}$$

with $F_R(\cdot)$ standing for the cumulative distribution function (CDF) of the capacity.

Unfortunately, this approach is extremely sensitive to the modelling of *R* and/or *S*. Outside of Reliability Theory, considerable progress has been made in recent years in the accurate calculation of characteristic traffic load effects on bridges, refer for instance to Bailey [2], Nowak [3], O'Connor [4], Grave [5], Caprani [6] or Jacob [7] for further reading. These characteristic load effect levels are found from an extrapolation of sample values obtained from measurement on site, to the required return period. Recent research work on the probabilistic analysis of highway bridge traffic loading by Caprani [6] has shown that these extreme values can be derived from Generalized Extreme Value (GEV) probability distributions.

Although this work does not facilitate calculations of probability of failure, as it provides one characteristic value, this characteristic value is derived from the lifetime distribution of the loading.

PREDICTIVE LIKELIHOOD

It has been common practice in engineering to define the characteristic value of a magnitude as this value of the considered stochastic variable with a fixed probability of exceedance. For example, the Eurocode for bridge loading [8] defines the characteristic value of the loading as that which is expected to have a 10% probability of exceedance in 100 years. This is usually expressed as a 1000-year return period.

Caprani [9] has shown that from given loading data not only a characteristic value can be derived, but the complete lifetime extreme load effect distribution.

If the PDF of load effect for an individual crossing were known, then the PDF for the lifetime maximum load effect could be calculated and the probability of failure found from Eq. (2). However, the PDFs are generally not known. The concept developed in this paper is to use Predictive Likelihood (PL) to estimate this distribution of lifetime maximum loading and to use it to estimate the lifetime probability of failure.

PL ranks all possible predictions by their joint likelihood given the observed data. The mathematical concept behind PL, as shown by Pawitan [10] relies on the maximization of the joint likelihood of a set of data and a fixed predictand obtaining a lifetime distribution of the considered effect (load effect in this case).

$$L_P(z \mid y) = \sup_{\theta} = L_y(\theta \mid y) \cdot L_z(\theta \mid z) \qquad \text{Eq. (3)}$$

where $L_P(z|y)$ is the maximized joint likelihood of data and predictand, $L_y(\theta|y)$ represents the likelihood of the data and $L_z(\theta|z)$ is the likelihood of the predictand. Finally θ represents the vector of parameters of the statistical distribution that represents the data and predictand.

Eq. (3) is termed Fisherian predictive likelihood after Fisher [11].

By maximizing this joint likelihood for all posible predictands, the complete statistical distribution of the lifetime extreme load effect may be determined, as already stated. However, due to practical reasons, usually only a discrete set of predictands is considered. The characteristic value has been defined as the value from this PDF with a 10% probability of exceedanc. This approach provides considerably more information that an extrapolation which gives just one estimate of the characteristic value. On the other hand, it opens the possibility to the use of Reliability Theory to compute the probability of failure.

Consequently, the characteristic lifetime of an engineering structure could be defined as the return period for which the ‘sum’ of the failure probabilities over all the cases of resistance for which the load exceeds the resistance does not exceed an assumed value.

PROBABILITY OF FAILURE

The probability of failure is obtained as the sum of the failure of probabilities over all the cases of resistance for which the load effect exceeds the resistance. This sum can be mathematically expressed by means of the convolution integral of the product of the PDF of the loading and the CDF of the resistance as defined in Eq. (2). The solution to this integral, can be found accurately using numerical techniques. The inaccuracy of the obtained probability of failure derives only from the modelling of the stochastic variables, as already stated.

This approach is very useful for the assessment of existing structures as both PDFs for loading and resistance can be determined at a certain time and consequently the risk of failure can be determined.

Once obtained this probability engineering criterion has to be adopted in order to determine if this is probability is acceptable or not. Melchers [1] or COST 345 report [12] propose acceptable risks in society for different events.

DISTRIBUTION OF THE MAXIMUM OF SAMPLE SETS

Given the PDF of any load effect for an individual crossing, we aim to determine the PDF for the lifetime maximum load effect. The distribution of a maximum of *n* sample repetitions of independent identically distributed (iid) variables is defined by Castillo [13] as:

$$F_Y(y) = P[(X_1 \leq y) \cap (X_2 \leq y) \cap ... \cap (X_n \leq y)] = \{F_X(x)\}^n \qquad \text{Eq. (4)}$$

with $Y = \max\{X_1, X_2, ..., X_n\}$

where $F_X(x)$ is the the common CDF of the variables X_i and $F_n(y)$ the corresponding CDF of Y.

The same idea can be applied to determine the distribution of the minimum of a set of iid variables

APPLICATION

For the purposes of this paper, we will consider a problem where both distributions of loading and resistance are normal. Let us assume a 30 m span (L) bridge loaded with a central load Q normally distributed $N \sim (\mu_Q, \sigma_Q)$ having mean $\mu_Q = 506$ kN and variance $\sigma_Q^2 = 2844$ (kN)^2. The bending capacity of this bridge follows a normal distribution $N \sim (\mu_R, \sigma_R)$ with a mean strength $\mu_R = 7500$ kNm and variance $\sigma_R^2 = 360000$ (kNm)^2.

From basic structural theory it can be shown that the applied bending moment (the load effect S) at the centre of the beam is given by:

$$S = \frac{Q\,l}{4}$$

The mean and standard deviation for the loading effect, the bending moment in the central section of the bridge can be computed as

$$\mu_S = \frac{30}{4}\mu_Q = 3800 \text{ kNm} \;\; ; \;\; \sigma_S^2 = \left(\frac{30}{4}\right)^2 \sigma_Q^2 = 160000 \; (\text{kNm})^2$$

with S following a normal distribution N~(μ_S, σ_S).

Let us now assume that this load effect represents the peak value from an individual traffic load crossing the bridge. We define the lifetime as 100 years, and consider 2000 crossings (trucks) per day, 250 working days per year. The number of considered events is thus 2000 x 250 x 100 = $50 \cdot 10^6$. The problem consists on determining a value to be the probability of any of the $50 \cdot 10^6$ events exceeding the bridge capacity.

The distribution of the maximum of these $50 \cdot 10^6$ repetitions is then, as given by Eq. (4).

$$F_n(y) = \{F_X(x)\}^{50 \cdot 1e6}$$

where for our example, all $F_X(x)$ are normal distributions and n=$50 \cdot 10^6$.

The probability distributions for individual load effect and capacity as well as for their extreme (maximum in lifetime) distributions are presented in Fig. 1.

Fig. 1 PDF for load effect $f_S(\cdot)$, capacity $f_R(\cdot)$, and lifetime maximum loading.

While the amount of overlap of the distribution of $f_R(\cdot)$ capacity and of the distribution of the maximum of $f_S(\cdot)$ can be taken as a rough indicator of the probability of load exceeding capacity (i.e., probability of failure), the exact probability can be derived from Eq (2)

$$p^*_f = P(R - S \leq 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{s \geq r} f_R(r) f_S^{max}(s)\, dr ds = P(Z \leq 0)$$

where p^*_f represents the lifetime probability of failure and f_s^{max} is the PDF of the maximum of the loading.

For this example, Monte Carlo Simulation is used to generate 3 set of 200 samples load effects from the original distribution. Typically, there would be many more values as there would be several days of measurement or simulations of load effect. Predictive Likelihood estimates the distribution of lifetime maximum load effect, given the sample of measured or simulated values.

The PL analysis of the 3 data sets is shown in figure 2. As can be seen, this results are not specially sensitive to the different samples.

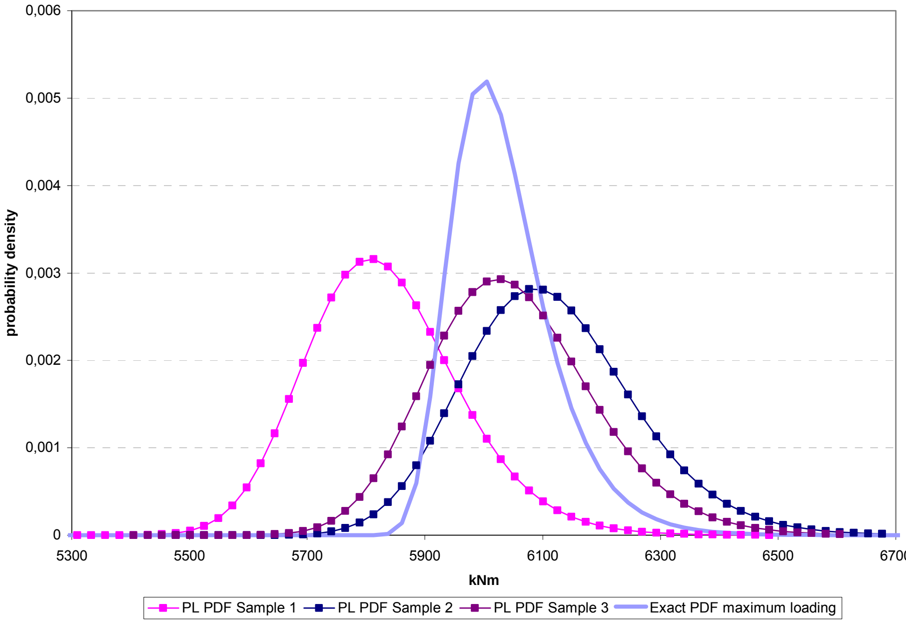

Fig. 2 Predictive likelihood PDFs derived for the 3 samples of loading, together with the exact PDF distribution of the maximum of the loading.

Figure 3 shows one of the predictive likelihood lifetime distributions obtained in the previous analysis together with the corresponding exact lifetime distribution for the loading derived from Eq (3). Parent PDF for loading as well as PDF for capacity are also displayed in the same figure.

Finally, for this example, the p_f^* as defined by Eq (2) is computed for the exact solution and for the predictive likelihood approximation of the maximum of the loading and the results for each of the samples and the average of the 3 samples are presented in table 1.

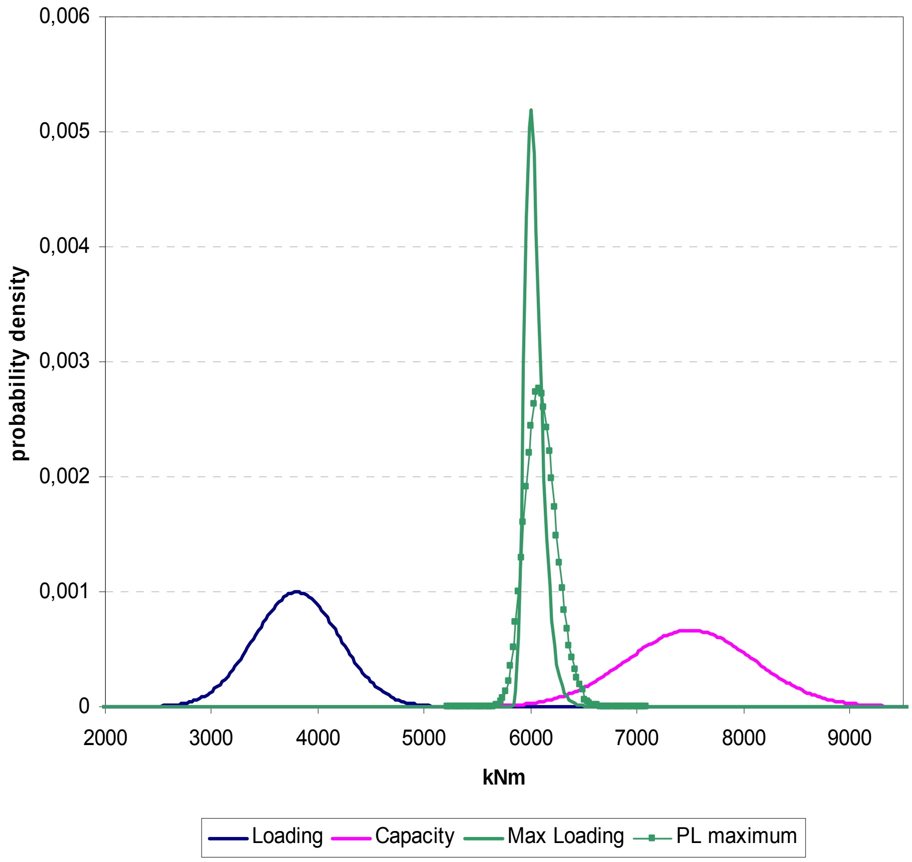

Fig. 3 Lifetime distributions for the exact solution and the PL approximation for loading effect.

It is remarkable to note that although the PL approximation is not specially sensitive to the samples, the probability of failure is. However, the average of the 3 probabilities of failures represents a better approximation to the calculated exact probability of failure that any of the samples.

Table 1 Probabilities of failure for exact solution and PL approximations.

	Probability of failure p_f*
Exact Solution	$3.29 \cdot 10^{-4}$
Sample 1	$1.39 \cdot 10^{-4}$
Sample 2	$5.20 \cdot 10^{-4}$
Sample 3	$3.91 \cdot 10^{-4}$
Average of 3 samples	$3.50 \cdot 10^{-4}$

CONCLUSIONS

In this paper predictive likelihood has been shown as an appropriate tool to determine lifetime probabilities of failure of existing structures. A simple application is considered, where the exact result is known, and can be compared to the results obtained with the proposed methodology.The lifetime distribution for load effect is obtained, using predictive likelihood. Following this, numerical integration is applied to compute the lifetime probability of failure.

The approximate predictive probability of failure has been found sensitive to the discretization of the lifetime distributions obtained using predictive likelihood as shown in table 1. Therefore it is recommended that a reasonably high number of points are used to fit the discrete lifetime distribution.

While the distributions for the exact PDF of the maximum of the loading and the PL approximation are significantly different as shown in Figure 2, it must be remembered that it is the result of an extrapolation from 200 samples to $50 \cdot 10^6$ events. At the same time higher number of samples in each set will lead to less disperse PL approximations, and consequently, more accurate results.

In the example, although only 200 outcomes were considered as the basis of the each of the lifetime distributions of loading effect approximated using PL, it has been show that predictive likelihood has a great potential to be used to accurately compute the lifetime probability of failure of existing structures.

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